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LEGENDRE AND CHEBYSHEV POLYNOMIALS ON A HYPERELLIPTIC CURVE: AN ELEMENTARY ACCOUNT

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1. Introduction

Let $p(x) \in \mathbb{C}[x]$ satisfy $p(0) \neq 0$, let $A = \mathbb{C}[x^{\pm 1}, u \mid u^2 = p(x)]$ be the coordinate ring of the affine hyperelliptic curve $u^2 = p(x)$ with the fibre over $x = 0$ removed, and let $\partial = u\,\frac{d}{dx}$; when the roots of p are simple, $\mathrm{Der}(A) = A\partial$ [8]. Two invariants of this data carry orthogonal polynomials. The first is the quotient $A/\partial A$, whose reduction to a basis is performed by the three-term recurrence of the Legendre polynomials evaluated at the branch parameter; this is equation (2.22) of Cox and Zhao [10]. The second is the group of units $A^\times$, determined for the quartic family in [8, Theorem 13] and described in [10, Theorem 5] by pairs (u_n, v_{n-1}) satisfying a polynomial Pell equation, of which the Chebyshev pairs (T_n, U_{n-1}) are the distinguished instance.

Both statements are proved in [10] inside a study of the universal central extension of $\mathrm{Der}(A)$, for general branch data, and the mechanism producing them is not separable from that setting there. It is separable, and it is elementary. The purpose of this account is to carry it out.

We fix the three smallest families,

$$p(x) = x - a, \qquad p(x) = x^2 - 2ax + 1, \qquad p(x) = x^4 - 2ax^2 + 1,$$

and compute $A/\partial A$ for each from the two formulas $\partial(x^n) = nx^{n-1}u$ and $\partial(x^n u) = nx^{n-1}p + \frac{1}{2}x^n p'$, which is the whole of the apparatus. The three dimensions themselves are instances of a closed formula: for p of degree n with simple nonzero roots, $\dim A/\partial A = n+1$. On the differential side this is Bremner [3, Theorem 3.4], who exhibits the basis $x^{-1}dx$, $x^{-1}u\,dx$, …, $x^{-n}u\,dx$ of Ω^1_A/dA; on the side of $A/\partial A$ it is [8, Theorem 9], quoted as [10, Theorem 2]. What the computation below supplies is the reduction that produces the number, class by class, and the recurrence that performs it, which for the quadratic family is Bonnet's. The dimension is then the number of classes the recurrence fails to determine from their neighbours: one for each integer at which the leading or the trailing coefficient vanishes and the reduction chain breaks, together with the free initial values of a chain that runs through all of $\mathbb{Z}$ unbroken. Nothing below uses Lie theory, cohomology, or the theory of orthogonal polynomials beyond the definitions.

Sections 3 to 5 carry out the three computations, and §6 treats the degenerate parameters $a = \pm 1$, at which p acquires a repeated root and A ceases to be a domain. One expects the dimension to drop there. It does not: the leading and trailing coefficients of the recurrence are $n+1$ and n, which vanish at the same two integers for every a, so the chain breaks in the same places and $\dim A/\partial A$ is 3 and 5 as in the smooth case (Propositions 6.2 and 6.3). We have not found this recorded. What does degenerate is the geometric side, and §6 says where.

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Section 7 recovers the same three dimensions from the topology of the curve, through the classical count $\dim \Omega^1_A/dA = 2g+r-1$, and exhibits the isomorphism $[f] \mapsto [f\,dx/u]$ responsible for the agreement. The puncture count is the step at which this calculation is usually got wrong, and Lemma 7.2 states it: $\operatorname{Spec} A$ omits the fibre over $x=\infty$ as well as the fibre over $x=0$, because x itself lies in A.

Sections 8 and 9 identify the two classical families. The second reads [10, Theorem 5] in the ring $\mathbb{C}[a, w \mid w^2 = a^2-1]$ generated by the branch parameter, where the Pell equation is the classical one and the unit $\varepsilon = a + w$ is fundamental. Placing the two families side by side puts one polynomial in two roles: the $1-2az+z^2$ that carries the Legendre generating function as $(1-2az+z^2)^{-1/2}$ carries the Chebyshev one as $(1-2az+z^2)^{-1}$, so that $U_n(a) = \sum_{k=0}^{n} P_k(a)P_{n-k}(a)$ (Proposition 9.3). That identity is classical. What we have not found stated is that its two sides come from the same curve, once through its defining equation and once through the norm form of the Pell equation for its parameter.

Section 10 states a case we could not settle.

For the Lie-theoretic setting from which these quotients come we refer to Kassel and Loday [13] and Kassel [12] for the general theory of universal central extensions and Kähler differentials, to Krichever and Novikov [14] and Schlichenmaier [17] for the algebras attached to a curve, and to [5, 8, 9, 7] and [1, 16] for the hyperelliptic and superelliptic cases. For the classical polynomials we follow Szegő [18]; Chihara [4], Andrews, Askey and Roy [2] and Ismail [11] are standard alternatives.

2. The ring, the derivation, and the two formulas

Definition 2.1. Let $p(x) \in \mathbb{C}[x]$ with $p(0) \neq 0$. Set

$$A = A_p := \mathbb{C}[x^{\pm 1}, u]/(u^2 - p(x)),$$

so that $A = \mathbb{C}[x^{\pm1}] \oplus \mathbb{C}[x^{\pm1}]\,u$ as a vector space, and let $\partial := u\,\frac{d}{dx}$.

Differentiating $u^2 = p(x)$ gives $2u\,\frac{du}{dx} = p'(x)$, whence $\partial(u) = u\,\frac{du}{dx} = \frac{1}{2}p'(x)$, and with the Leibniz rule this yields the two formulas on which everything below rests: for every $n \in \mathbb{Z}$,

$$\partial(x^n) = n\,x^{n-1}u, \tag{1}$$

$$\partial(x^n u) = n\,x^{n-1}p(x) + \tfrac{1}{2}\,x^n\,p'(x), \tag{2}$$

the first at once and the second from $\partial(x^n u) = nx^{n-1}u^2 + x^n\,\partial(u)$ together with $u^2 = p(x)$.

The images (1) lie in $\mathbb{C}[x^{\pm1}]u$ and the images (2) lie in $\mathbb{C}[x^{\pm1}]$, so the quotient splits,

$$A/\partial A \;=\; \big(\mathbb{C}[x^{\pm1}]/\langle \partial(x^n u)\rangle\big) \;\oplus\; \big(\mathbb{C}[x^{\pm1}]u/\langle \partial(x^n)\rangle\big), \tag{3}$$

and the second summand is independent of p.

Lemma 2.2. *$\mathbb{C}[x^{\pm1}]u/\langle \partial(x^n) : n \in \mathbb{Z}\rangle$ is one-dimensional, spanned by the class of $x^{-1}u$.*

Proof. By (1), $\partial(x^n) = n\,x^{n-1}u$. As n ranges over $\mathbb{Z}$ the exponents $n-1$ give every integer except -1, so the span of the $\partial(x^n)$ is $\{x^k u : k \neq -1\}$ and the only surviving class is that of $x^{-1}u$. □

All the content is therefore in the first summand of (3), which depends on p through (2).

3. The linear case

For $p(x) = x - a$ with $a \neq 0$ we have $p'(x) = 1$, and (2) reads

$$\partial(x^n u) = \big(n + \tfrac{1}{2}\big)x^n - a\,n\,x^{n-1}. \tag{4}$$

This is not a hyperelliptic curve, the genus being zero and p linear, and we treat it first because the mechanism is visible in two lines.

Proposition 3.1. *For $p(x) = x - a$ with $a \neq 0$, $\dim_{\mathbb{C}} A/\partial A = 2$, with basis $\{[x^{-1}], [x^{-1}u]\}$. For $a = 0$, $\dim_{\mathbb{C}} A/\partial A = 1$, spanned by $[x^{-1}u]$.*

Proof. Setting (4) to zero gives

$$\left(n + \tfrac{1}{2}\right)[x^n] = a\,n\,[x^{n-1}], \qquad n \in \mathbb{Z}. \tag{5}$$

The coefficient $n + \frac{1}{2}$ never vanishes on $\mathbb{Z}$, so (5) determines $[x^n]$ from $[x^{n-1}]$ at every n; at $n = 0$ it reads $\frac{1}{2}[1] = 0$, so $[1] = 0$ and with it every $[x^n]$, $n \geq 0$. Downwards, solving for $[x^{n-1}]$ requires dividing by an, which fails at $n = 0$; the chain is broken there, $[x^{-1}]$ survives, and every $[x^{-n}]$, $n \geq 2$, is a multiple of it. With Lemma 2.2 this gives the stated basis. For $a = 0$, (5) degenerates to $[x^n] = 0$ for all n. □

Two integers carry the dimension here, $n = 0$ in each of (1) and (2), so the count of breaks is the answer. That is an accident of the short recurrence. From §5 on, the recurrence splits by parity, and one of the two halves runs through all of $\mathbb{Z}$ without breaking anywhere and still contributes two classes. What carries over is the method: locate the breaks, then count what the chains leave behind.

4. The quadratic case

Let $p(x) = x^2 - 2ax + 1$, of discriminant $4(a^2 - 1)$, so that p has simple roots exactly when $a \neq \pm 1$. Then $p'(x) = 2x - 2a$ and (2) gives

$$\partial(x^n u) = (n+1)\,x^{n+1} - a(2n+1)\,x^n + n\,x^{n-1}, \tag{6}$$

which is the relation of [10, eq. (2.22)].

Proposition 4.1. *Let $a \neq \pm 1$. Then $\dim_{\mathbb{C}} A/\partial A = 3$, with basis*

$$\omega_0 = [x^{-1}u], \qquad \omega_1 = [x^{-1}], \qquad \omega_2 = [1].$$

Proof. Write $p_n = [x^n]$. Setting (6) to zero gives

$$(n+1)\,p_{n+1} - a(2n+1)\,p_n + n\,p_{n-1} = 0, \qquad n \in \mathbb{Z}. \tag{7}$$

The leading coefficient $n + 1$ vanishes only at $n = -1$, the trailing coefficient n only at $n = 0$. Upwards from $n = 0$ one gets $p_1 = ap_0$, then $2p_2 = 3ap_1 - p_0$, and inductively every p_n with $n \geq 1$ is a multiple of p_0. Downwards from $n = -1$ the leading term drops out and the relation reads $a\,p_{-1} = p_{-2}$, so inductively every p_{-n} with $n \geq 2$ is a multiple of p_{-1}. The two chains are not joined, since the only relation that could join them is the one at $n = -1$, in which p_0 does not occur. With Lemma 2.2 the dimension is $1 + 1 + 1$. □

Example 4.2. At $a = 3$, (7) gives $p_1 = 3p_0$, $p_2 = 13\,p_0$, $p_3 = 63\,p_0$.

5. The quartic case

Let $p(x) = x^4 - 2ax^2 + 1$, again with simple roots exactly for $a \neq \pm 1$. Here $p'(x) = 4x^3 - 4ax$ and (2) gives

$$\partial(x^n u) = (n+2)\,x^{n+3} - 2a(n+1)\,x^{n+1} + n\,x^{n-1}. \tag{8}$$

The exponents differ by 2, so the even and the odd powers of x never interact and the first summand of (3) splits again, into an *even sector* and an *odd sector*.

Proposition 5.1. *Let $a \neq \pm 1$. For $p(x) = x^4 - 2ax^2 + 1$, $\dim_{\mathbb{C}} A/\partial A = 5$, with basis the classes of 1, x^2, x, x^{-1}, $x^{-1}u$.*

Proof. Setting (8) to zero,

$$(n+2)\,p_{n+3} - 2a(n+1)\,p_{n+1} + n\,p_{n-1} = 0, \tag{9}$$

with coefficients vanishing at $n=-2$, $n=-1$, $n=0$ respectively.

Taking n odd makes all three exponents even. Upwards from $n=1$, every p_{2k} with $k\geq 2$ reduces to p_0 and p_2. At $n=-1$ the middle coefficient vanishes and the relation reads $p_2 - p_{-2} = 0$; at $n=-3$ it reads $-p_0 + 4a\,p_{-2} - 3p_{-4} = 0$, and inductively every p_{-2k} reduces. The even sector is spanned by p_0, p_2.

Taking n even makes all three exponents odd. At $n=0$ the trailing coefficient vanishes and the relation reads $p_3 = a\,p_1$, so every p_{2k+1} with $k\geq 1$ is a multiple of p_1. At $n=-2$ the leading coefficient vanishes and the relation reads $p_{-3} = a\,p_{-1}$, so every $p_{-(2k+1)}$ with $k\geq 1$ reduces to p_{-1}. The two chains are not joined, as in Proposition 4.1. The odd sector is spanned by p_1, p_{-1}.

With Lemma 2.2, $2+2+1=5$. □

6. The degenerate parameters

At $a=1$ the quadratic $p(x)=(x-1)^2$ has a repeated root, the relation $u^2=(x-1)^2$ factors as $(u-(x-1))(u+(x-1))=0$, and A is no longer a domain: the curve is two copies of $\mathbb{C}^\times$ meeting at $(x,u)=(1,0)$.

Lemma 6.1. *For $p(x)=(x-1)^2$ and $B=\mathbb{C}[x^{\pm 1}]$,*

$$A \;\cong\; B\times_{\mathbb{C}} B = \{(f,g)\in B\times B : f(1)=g(1)\},$$

the fibre product over evaluation at $x=1$, and under this isomorphism ∂ becomes $(\partial_1, -\partial_1)$ with $\partial_1 = (x-1)\frac{d}{dx}$.

Proof. Send $h\in A$ to its pair of restrictions to the branches $u=\pm(x-1)$. The image is the set of pairs agreeing at $x=1$: if $f(1)=g(1)$ then $(x-1)\mid(f-g)$, so the pair can be written in exactly one way as $\big(h_0+(x-1)k,\ h_0-(x-1)k\big)$, which is the image of $h_0+ku\in A$. Since A is free over B on $1,u$, the map is an isomorphism. On $u=x-1$ the derivation becomes ∂_1, on $u=-(x-1)$ its negative. □

The identification is a fibre product and not a direct product, which would describe two disjoint curves. The ring degenerates and the curve degenerates; the quotient does not.

Proposition 6.2. *For $p(x)=(x-1)^2$, $\dim_{\mathbb{C}} A/\partial A = 3$, with the basis of Proposition 4.1.*

Proof. No step of Proposition 4.1 used $a\neq\pm 1$. Formula (2) still yields (6), now at $a=1$, and the coefficients $n+1$ and n vanish at $n=-1$ and $n=0$ independently of a, so the two breaks are in the same places and the argument applies verbatim. □

Proposition 6.3. *For $p(x)=(x^2-1)^2$, $\dim_{\mathbb{C}} A/\partial A = 5$.*

Proof. The coefficients $n+2$, $-2a(n+1)$, n of (9) vanish at $n=-2,-1,0$ independently of a, and Proposition 5.1 applies at $a=1$ unchanged. □

The quotient $A/\partial A$ is insensitive to the degeneration because the recurrence is a polynomial identity in a whose break structure is fixed by the integer coefficients alone. The geometry is not insensitive. At $a=\pm 1$ the curve acquires a node, the smooth-curve count of Theorem 7.1 below does not apply, and the isomorphism of Proposition 7.4 fails, since dx/u ceases to generate Ω^1_A at the node. We claim nothing about Ω^1_A/dA at the node; computing it directly would measure how far the two sides separate there.

One route to the two propositions above deserves a warning, because it is the natural one and gives the wrong answer. Instead of running the recurrence at $a=1$ one may pass to the

normalization, replace A by its branches and correct by the first Betti number of the dual graph. On a single branch $\partial_1(x^n) = n(x^n - x^{n-1})$ identifies $[x^n]$ with $[x^{n-1}]$ *only for* $n \neq 0$; the chain breaks at $n = 0$ as everywhere else, so $\dim B/\partial_1 B = 2$ rather than 1. Two branches give 4. The fibre product condition then costs exactly one: it does not shrink ∂A, because a constant may be added to either component without changing its image under ∂_1, so ∂A is all of $\partial_1 B \times \partial_1 B$, while A itself has codimension one in $B \times B$. The answer is again 3. Reading the identification as unconditional loses that break and costs one dimension.

7. The same dimensions from the topology of the curve

Let Ω^1_A be the A-module of Kähler differentials and dA its submodule of exact forms. The quotient Ω^1_A/dA is computed without any recurrence. On a curve there are no forms of degree two, so it is the first algebraic de Rham cohomology of $\operatorname{Spec} A$; it is also the first cyclic homology $HC_1(A)$ in the sense of [15], and it is in that shape that it enters the theory of central extensions, as the centre of the universal one [13, 12]. The count itself is topological and classical.

Theorem 7.1. *Let X be a smooth affine curve obtained from a compact Riemann surface of genus g by deleting $r \geq 1$ points. Then* $\dim_{\mathbb{C}} \Omega^1_{\mathbb{C}[X]}/d\,\mathbb{C}[X] = 2g + r - 1$.

Algebraic and topological de Rham cohomology agree on a smooth affine variety, and a genus-g surface with r punctures has the homotopy type of a wedge of $2g+r-1$ circles. Applying the theorem requires counting those punctures.

Lemma 7.2. *Let A be as in Definition 2.1 with* $\deg p = d$ *and p of simple roots, and let $\overline{X}$ be the smooth compactification of $u^2 = p(x)$. Then* $\operatorname{Spec} A = \overline{X} \setminus S$, *where S consists of the points over $x = 0$ together with those over $x = \infty$, and*

$$r = \#S = 2 + \begin{cases} 2, & d \text{ even}, \\ 1, & d \text{ odd}. \end{cases}$$

Proof. A inverts x, so the fibre over $x = 0$ is deleted, and $p(0) \neq 0$ says the double cover is unramified there, which gives two points. A also contains x, so x is regular on $\operatorname{Spec} A$ and the fibre over $x = \infty$ is deleted as well. For d even, $v = u/x^{d/2}$ has two limiting values at infinity, whose squares tend to the leading coefficient of p, so there are two points over $x = \infty$; for d odd that point is a branch point and there is one. □

The clause about $x = \infty$ is the one that is missed by the argument that A inverts x and therefore deletes only the fibre over 0.

Proposition 7.3. *For the three families, with the parameter generic:*

$p(x)$	g	r	$2g + r - 1$	$\dim A/\partial A$
$x - a$, $a \neq 0$	0	3	2	2
$x^2 - 2ax + 1$, $a \neq \pm 1$	0	4	3	3
$x^4 - 2ax^2 + 1$, $a \neq \pm 1$	1	4	5	5

Proof. The genus of $u^2 = p(x)$ with p of degree d and simple roots is $\lfloor (d-1)/2 \rfloor$; the punctures are Lemma 7.2; the last column is Propositions 3.1, 4.1 and 5.1. □

Proposition 7.4. *The assignment $\Phi\colon A/\partial A \to \Omega^1_A/dA$, $[f] \mapsto [f\,dx/u]$, is a well-defined isomorphism of vector spaces.*

Proof. Differentiating $u^2 = p(x)$ gives $2u\,du = p'(x)\,dx$, so $dx/u = 2\,du/p'(x)$ away from the branch points, and the two expressions patch to a generator of Ω^1_A as an A-module. For $h \in A$,

$$\partial(h)\,\frac{dx}{u} = u\,\frac{dh}{dx} \cdot \frac{dx}{u} = dh \in dA,$$

so ∂A maps into dA and Φ is well defined. It is surjective because dx/u generates. For injectivity, write $h = f + gu$ with $f, g \in \mathbb{C}[x^{\pm 1}]$ and compute both sides:

$$dh = \big(f'u + g'p + \tfrac{1}{2}gp'\big)\frac{dx}{u}, \qquad \partial(h) = f'u + g'p + \tfrac{1}{2}gp',$$

the second by (1) and (2). The two agree, so dA is exactly the image of ∂A under the free generator dx/u, and Φ is injective. □

Proposition 7.4 identifies the two invariants once and for all, so the basis of Ω^1_A/dA recorded in [3, Theorem 3.4] transports to $A/\partial A$: the class $x^{-1}dx$ corresponds to $[x^{-1}u]$ and $x^{-i}u\,dx$ to $[x^{-i}p]$, for $i = 1, \dots, \deg p$.

8. The Legendre family

Recall Bonnet's recurrence for the Legendre polynomials, $(n+1)P_{n+1}(t) = (2n+1)\,t\,P_n(t) - n\,P_{n-1}(t)$, with $P_0 = 1$ and $P_1(t) = t$ [18, §4.7]. The following is [10, eq. (2.22)], in the notation of §4.

Theorem 8.1 (Cox–Zhao)**.** *For $p(x) = x^2 - 2ax + 1$ with $a \neq \pm 1$,*

$$[x^{\,k}] = P_k(a)\,\omega_2 \quad (k \geq 0), \qquad [x^{-k}] = P_{k-1}(a)\,\omega_1 \quad (k \geq 1).$$

Proof. Written homogeneously, Bonnet's recurrence is $(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}(t) = 0$, whose coefficients are those of (7) under $t \leftrightarrow a$. The initial data agree: $p_0 = \omega_2$ and $p_1 = a\omega_2$ from the $n = 0$ instance of (7), against $P_0(a) = 1$ and $P_1(a) = a$. A three-term recurrence with nonvanishing leading coefficient is determined by two initial values, so the sequences agree for $k \geq 0$. The negative branch is the same argument run downwards from $p_{-1} = \omega_1$, using $a\,p_{-1} = p_{-2}$. □

The numbers $1, 3, 13, 63$ of Example 4.2 are $P_0(3), \dots, P_3(3)$.

Theorem 8.2. *Let $F(z) = \sum_{n\geq 0} P_n(a)z^n$. Then*

$$(1 - 2az + z^2)\,F'(z) + (z - a)\,F(z) = 0, \tag{10}$$

and the solution with $F(0) = 1$ is $F(z) = (1 - 2az + z^2)^{-1/2}$.

Proof. Multiplying (7) by z^n and summing over $n \geq 0$ turns the three terms into F', $-a(2zF' + F)$ and $z^2F' + zF$, giving (10). The equation is separable, with $dF/F = -\frac{1}{2}\,d[\log(1 - 2az + z^2)]$, so $F = C(1 - 2az + z^2)^{-1/2}$ and $F(0) = 1$ forces $C = 1$. □

The polynomial $1 - 2az + z^2$ here is the reciprocal of the curve polynomial $x^2 - 2ax + 1$, and since that polynomial is palindromic it is its own reciprocal, so the two are the same polynomial in different variables. The curve chooses the generating function, and the generating function chooses the family.

Proposition 8.3. *For $p(x) = x^4 - 2ax^2 + 1$, $[x^{2k+1}] = P_k(a)\,[x]$ for $k \geq 0$, with $\sum_{k\geq 0} P_k(a)z^{2k+1} = z(1 - 2az^2 + z^4)^{-1/2}$.*

Proof. Restricting (9) to $n = 2k$ even and writing $q_k = [x^{2k+1}]$ gives $(2k+2)q_{k+1} - 2a(2k+1)q_k + 2k\,q_{k-1} = 0$, which is (7) after dividing by 2. The generating function is that of Theorem 8.2 with z replaced by z^2 and multiplied by z. □

9. The Chebyshev family and the Pell equation

Take the quartic in the normalisation $p(t) = (t^4 - 2\beta t^2 + 1)/(\beta^2 - 1)$, the case of Date, Jimbo, Kashiwara and Miwa, whose algebras are studied in [6]. Its unit group was determined by Cox, Guo, Lu and Zhao [8, Theorem 13]: $A^\times \cong \mathbb{C}^\times \times \mathbb{Z}^3$. Writing $u_n + v_{n-1}\sqrt{p} = \lambda^n$ for a distinguished unit λ, the pair (u_n, v_{n-1}) obeys $y_{n+2} - 2q\,y_{n+1} + y_n = 0$ with $q = (t^2 - \beta)/\sqrt{\beta^2 - 1}$, which is the Chebyshev recursion in q; this is [10, Theorem 5].

The same statement in the ring generated by the branch parameter is the classical Pell equation, and it is worth seeing on its own, because the roots of $x^2 - 2ax + 1$ are $a \pm \sqrt{a^2 - 1}$, of product 1. Set

$$R := \mathbb{C}[a, w]/(w^2 - (a^2 - 1)),$$

a ring of the shape of Definition 2.1 one degree lower, with the branch parameter as its variable. Both statements below are classical in this ring: the first is the polynomial Pell equation and the second its solution by Chebyshev pairs. We give the arguments because they are short and because they are what [10, Theorem 5] becomes here.

Proposition 9.1. *An element $T + Uw \in R$ with $T, U \in \mathbb{C}[a]$ is a unit if and only if $T^2 - (a^2-1)U^2$ is a nonzero constant. The elements of norm 1 form a group isomorphic to $\mathbb{Z} \times \{\pm 1\}$, generated by $\varepsilon := a + w$ and -1.*

Proof. The norm $N(T + Uw) = T^2 - (a^2 - 1)U^2$ is multiplicative with values in $\mathbb{C}[a]$, and an element is a unit exactly when its norm is a unit of $\mathbb{C}[a]$. Now $N(\varepsilon) = a^2 - (a^2 - 1) = 1$, so $\varepsilon^{-1} = a - w$. Conversely, let $N(T + Uw) = 1$ with $U \neq 0$ and $m = \deg U$. Then $\deg((a^2 - 1)U^2) = 2m + 2 = \deg T^2$, so $\deg T = m + 1$, and comparing leading coefficients gives $t^2 = u^2$ for the leading coefficients t of T and u of U. Multiplying by ε or by ε^{-1}, whichever cancels the two leading terms, yields a norm-one element whose w-component has degree at most $m - 1$. The descent terminates at $U = 0$, $T = \pm 1$. □

Theorem 9.2 (Cox–Zhao, read in R)**.** *For $n \geq 1$, $\varepsilon^{\,n} = T_n(a) + U_{n-1}(a)\,w$, where T_n and U_n are the Chebyshev polynomials of the first and second kind. Equivalently $T_n(a)^2 - (a^2-1)U_{n-1}(a)^2 = 1$.*

Proof. Multiplying by $\varepsilon = a + w$ and using $w^2 = a^2 - 1$,

$$\varepsilon^{\,n+1} = \big(aT_n + (a^2 - 1)U_{n-1}\big) + \big(T_n + aU_{n-1}\big)w,$$

and the identities $T_{n+1} = aT_n + (a^2-1)U_{n-1}$, $U_n = T_n + aU_{n-1}$ [2, §5.1] identify the components; the base case $n = 1$ is $T_1(a) = a$, $U_0(a) = 1$. The Pell identity is $N(\varepsilon^n) = N(\varepsilon)^n = 1$. □

Theorem 9.2 is [10, Theorem 5] read in R, where q becomes the variable a and the recursion becomes the classical one. Placing it beside Theorem 8.2 puts the two generating functions side by side.

Proposition 9.3. *Let $F(z) = \sum_{n\geq 0} P_n(a) z^n$ and $G(z) = \sum_{n\geq 0} U_n(a) z^n$. Then*

$$G(z) = \frac{1}{1 - 2az + z^2} = F(z)^2, \qquad \text{that is} \qquad U_n(a) = \sum_{k=0}^{n} P_k(a)\,P_{n-k}(a). \tag{11}$$

Proof. $G(z) = (1 - 2az + z^2)^{-1}$ is the standard generating function of the U_n [18, §4.7], and it is the square of $F(z) = (1 - 2az + z^2)^{-1/2}$ from Theorem 8.2. □

The identity (11) between the two generating functions is classical and belongs to the theory of the Gegenbauer family, of which both are members. We record its source here: the same polynomial $1 - 2az + z^2$ occurs as the defining equation of the curve, where its reciprocal square root reduces $A/\partial A$, and as the norm form of the Pell equation for the branch parameter, where

its reciprocal enumerates the units. The exponents $-\frac{1}{2}$ and -1 are the square root in the curve equation and the norm in the Pell equation.

10. A case we could not settle

The next family is $u^2 = x^N - 1$ with $N \geq 3$, whose branch points are the N-th roots of unity. Here $p'(x) = Nx^{N-1}$ and (2) gives

$$\partial(x^n u) = \Big(n + \tfrac{N}{2}\Big)x^{n+N-1} - n\,x^{n-1},$$

a two-term relation identifying $[x^{n+N-1}]$ with a multiple of $[x^{n-1}]$, in steps of N, except at $n = 0$ and, when N is even, at $n = -N/2$. Being two-term, it admits no orthogonal family in the sense of Favard's theorem [4, Theorem 4.4], and the reduction coefficients are products of linear factors.

Problem 10.1. Identify the family of polynomials in the branch parameter that performs the reduction in $A/\partial A$ for the deformed family $u^2 = x^N - 2ax^{\lfloor N/2 \rfloor} + 1$, in the sense that Theorem 8.1 does for $N = 2$, and settle $N = 3, 4, 5$.

Cox and Zhao treat the general branch data in [10], where associated Legendre polynomials appear; the question here is which family survives the reciprocal symmetry $p(x) = x^N p(1/x)$, which the deformed equation has exactly when N is even. For even N we expect a Chebyshev or Dickson family, on the grounds that §9 produced one from the same shape of equation one degree lower; for odd N the deformation is not self-reciprocal and we have no expectation to offer. We have no proof in either case and state this as an expectation only.

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